Answers to review questions from sections 3.1-3.7 and chapter 1

(1) Use the definition \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) of the derivative to find \( f'(x) \) when \( f(x) = x^{-1/2} \).

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x} \sqrt{x+h}}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{1}{\sqrt{x} \sqrt{x+h}} \cdot \frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{\sqrt{x} + \sqrt{x+h}}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{1}{\sqrt{x} \sqrt{x+h}} \cdot \frac{x - (x+h)}{\sqrt{x} + \sqrt{x+h}}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{1}{\sqrt{x} \sqrt{x+h}} \cdot \frac{-h}{\sqrt{x} + \sqrt{x+h}}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{x} \sqrt{x+h}} \cdot \frac{-1}{\sqrt{x} + \sqrt{x+h}}
\]

\[
= -\frac{1}{\sqrt{x} \sqrt{x}(2\sqrt{x})} = -\frac{1}{2x^{3/2}} = -\frac{1}{2x^{3/2}}
\]

(2) Use the definition \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) of the derivative to find \( f'(x) \) when \( f(x) = x^{-2} \).

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{(x+h)^2} \cdot \frac{1}{x} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{x^2 - (x + h)^2}{x(x + h)^2}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{x^2 - (x^2 + 2hx + h^2)}{x(x + h)^2}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-2hx - h^2}{x(x + h)^2}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-2x - h}{x(x + h)^2}
\]

\[
= \lim_{h \to 0} \frac{-2x}{x^2(x + h)^2}
\]

\[
= -\frac{2x}{x^2} = -2x^{-3}
\]

In questions (3)-(7), Find the derivative of the given function of \( x \).

(3) \[ \frac{\sin(x^2)}{(1 + e^{7x})^2} \]

\[
\frac{d}{dx} \frac{\sin(x^2)}{(1 + e^{7x})^2} = \left( \frac{\frac{d}{dx} \sin(x^2)}{1 + e^{7x}} \right) \left( \frac{1 + e^{7x})^2 - \sin(x^2) \frac{d}{dx} (1 + e^{7x})^2}{(1 + e^{7x})^4} \right)
\]

\[
= \frac{2x \cos(x^2)(1 + e^{7x})^2 - \sin(x^2)2(1 + e^{7x})7e^{7x}}{(1 + e^{7x})^4}
\]

\[
= \frac{2x \cos(x^2)(1 + e^{7x}) - \sin(x^2)2 \cdot 7e^{7x}}{(1 + e^{7x})^3}
\]
(4) \( \cot(x + x^5)\sqrt{x^2 + \tan^2 x} \)

\[
\frac{d}{dx} \left( \cot(x + x^5)\sqrt{x^2 + \tan^2 x} \right) \\
= \left( \frac{d}{dx} \cot(x + x^5) \right) \sqrt{x^2 + \tan^2 x} + \cot(x + x^5) \frac{d}{dx} (x^2 + \tan^2 x)^{1/2} \\
= - (1 + 5x^4) \csc^2(x + x^5)\sqrt{x^2 + \tan^2 x} \\
+ \cot(x + x^5)(1/2)(x^2 + \tan^2 x)^{-1/2}(2x + 2\tan x\sec^2 x)
\]

(5) \( \csc(x^3 + 2) \)

\[
\frac{d}{dx} \csc(x^3 + 2) = -\csc(x^3 + 2) \cot(x^3 + 2) \frac{d}{dx} (x^3 + 2) = -\csc(x^3 + 2) \cot(x^3 + 2)(3x^2)
\]

(6) \( \sec(e^{x^2+5}) \)

\[
\frac{d}{dx} \sec(e^{x^2+5}) = \sec(e^{x^2+5}) \tan(e^{x^2+5}) \frac{d}{dx} e^{x^2+5} = \sec(e^{x^2+5}) \tan(e^{x^2+5})(2e) e^{x^2+5}
\]

(7) \( \frac{1}{\sqrt{x^2 + \sec^2 x}} \)

\[
\frac{d}{dx} \left( \frac{1}{\sqrt{x^2 + \sec^2 x}} \right) = \frac{d}{dx} (x^2 + \sec^2 x)^{-1/2} = (-1/2)(x^2 + \sec^2 x)^{-3/2} \frac{d}{dx} (x^2 + \sec^2 x) \\
= (-1/2)(x^2 + \sec^2 x)^{-3/2}(2x + 2\sec x \sec x \tan x)
\]

In questions (8)-(11), Find the second derivative of the given function of \( x \).

(8) \( \tan(x^4 + 1) \)

The first derivative is

\[
\frac{d}{dx} \tan(x^4 + 1) = \sec^2(x^4 + 1) \frac{d}{dx} (x^4 + 1) = (\sec^2(x^4 + 1))(4x^3) = 4x^3 \sec^2(x^4 + 1)
\]

The second derivative is

\[
\frac{d}{dx} (4x^3 \sec^2(x^4 + 1)) = 12x^2 \sec^2(x^4 + 1) + 4x^3 2 \sec(x^4 + 1)(\sec(x^4 + 1) \tan(x^4 + 1))4x^3
\]

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\( \sqrt{2 + \cos x} \)

The first derivative is
\[
\frac{d}{dx}(2 + \cos x)^{1/2} = (1/2)(2 + \cos x)^{-1/2}(-\sin x) = (-1/2)(\sin x)(2 + \cos x)^{-1/2}
\]

The second derivative is
\[
\frac{d}{dx} \left( (-1/2)(\sin x)(2 + \cos x)^{-1/2} \right) \\
= (-1/2)(\cos x)(2 + \cos x)^{-1/2} + (-1/2)(\sin x)(-1/2)(2 + \cos x)^{-3/2}(-\sin x) \\
= - (1/2)(\cos x)(2 + \cos x)^{-1/2} - (1/4)(\sin^2 x)(2 + \cos x)^{-3/2}
\]

\( \sec(x^3) \)

The first derivative is
\[
\frac{d}{dx} \sec(x^3) = 3x^2 \sec(x^3) \tan(x^3)
\]

Using \((fgh)' = f'gh + fg'h + fgh'\) we find that the second derivative is
\[
\frac{d}{dx} (3x^2 \sec(x^3) \tan(x^3)) \\
= 6x \sec(x^3) \tan(x^3) + 3x^2(3x^2 \sec(x^3) \tan(x^3)) \tan(x^3) + 3x^2 \sec(x^3) \sec^2(x^3) 3x^2
\]

\( \frac{1}{5 + x^2} \)

The first derivative is
\[
\frac{d}{dx} \left( \frac{1}{5 + x^2} \right)^{-1} = (-1)(5 + x^2)^{-2}2x = -2x(5 + x^2)^{-2}
\]

The second derivative is
\[
\frac{d}{dx} \left( -2x(5 + x^2)^{-2} \right) = -2(5 + x^2)^{-2} - 2x(-2)(5 + x^2)^{-3}(2x) = -2(5 + x^2)^{-2} + 8x^2(5 + x^2)^{-3}
\]

(12) Find all points at which the tangent line of the function \( f(x) = x(1 + x^2)^{-2} \) is horizontal.

The derivative of the function is
\[
\frac{d}{dx} \left( x(1 + x^2)^{-2} \right) = (1 + x^2)^{-2} + x(-2)(1 + x^2)^{-3}2x = (1 + x^2)^{-2} - 4x^2(1 + x^2)^{-3} \\
= (1 + x^2 - 4x^2)(1 + x^2)^{-3} = (1 - 3x^2)(1 + x^2)^{-3}
\]

The derivative is zero when \( 1 - 3x^2 = 0 \), and this happens when \( x = 1/\sqrt{3} \) and \( x = -1/\sqrt{3} \).
(13) Find the equation for the line tangent to the function \( f(x) = \cos(2x) \) at the point \((\pi/6, f(\pi/6))\).

Using \( f'(x) = -2\sin(2x) \) we find \( f'(\pi/6) = -2\sin(\pi/3) = -2(\sqrt{3}/2) = -\sqrt{3} \). The equation of the tangent line is

\[
y - 1/2 = -\sqrt{3}(x - \pi/6)
\]

(14) Assume \( x \) is a number such that \( \tan x = 7 \) and \( \sin x < 0 \). Find \( \sec x \).

We know \( \sec^2 x = 1 + \tan^2 x = 1 + 7^2 = 50 \). This implies \( \sec x = \pm\sqrt{50} = \pm5\sqrt{2} \). Since \( \frac{\sin x}{\cos x} = \tan x = 7 > 0 \) and \( \sin x < 0 \), we conclude \( \cos x < 0 \), hence \( \sec x = \frac{1}{\cos x} < 0 \). This fact and \( \sec x = \pm5\sqrt{2} \) imply \( \sec x = -5\sqrt{2} \).

(15) Simplify \( \sin(\sin^{-1} x), \cos(\sin^{-1} x), \sec(\sin^{-1} x), \tan(\sin^{-1} x) \).

The definition of inverse function implies \( \sin(\sin^{-1} x) = x \). Since

\[
\cos^2(\sin^{-1} x) = 1 - \sin^2(\sin^{-1} x) = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2,
\]

we conclude \( \cos(\sin^{-1} x) = \pm\sqrt{1-x^2} \). But \( \sin^{-1} x \) is in the interval \([-\pi/2, \pi/2]\) and \( \cos \) is positive or zero on that interval. This implies \( \cos(\sin^{-1} x) = \sqrt{1-x^2} \). Now we know

\[
\sec(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}.
\]

Finally,

\[
\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}}.
\]

(16) Consider the function \( f(x) = \frac{x - 8}{1 + 7x} \). Find a formula for the inverse function \( f^{-1}(x) \).

The notation \( t = f^{-1}(x) \) gives \( x = f(t) = \frac{t - 8}{1 + 7t} \). When we solve for \( t \), we get \( t - 8 = x(1 + 7t) \), which is \( t - 8 = x + 7xt \), which is \( (1 - 7) t = x + 8 \), which is \( t = \frac{x + 8}{1 - 7x} \). We conclude \( f^{-1}(x) = t = \frac{x + 8}{1 - 7x} \).

(17) Solve for \( x \) in the equation \( e^{4x+3} = 2e^{3-x} \).

Dividing both sides of the equation \( e^{4x+3} = 2e^{3-x} \) by \( e^{3-x} \), we get \( e^{5x} = 2 \). This is \( 5x = \ln 2 \), which is \( x = \frac{\ln 2}{5} \).

(18) Solve for \( x \) in the equation \( \sqrt{e^{8x-6}} = e^x \).

This is \( (e^{8x-6})^{1/2} = e^x \), which is \( e^{(8x-6)(1/2)} = e^x \), which simplifies to \( e^{4x-3} = e^x \). Since the exponential function is a one-to-one function, we conclude \( 4x - 3 = x^2 \), which is \( x^2 - 4x + 3 = 0 \), which is \( (x - 1)(x - 3) = 0 \). The solutions are \( x = 1 \) and \( x = 3 \).
(19) Explain why the function \( f(x) = x^6 + 3 \) is not a one-to-one function.

We find \( f(-1) = (-1)^6 + 3 = 1^6 + 3 = f(1) \). We have \( f(-1) = f(1) \), but \(-1 \neq 1\). This tells us that the function \( f(x) \) is not one-to-one.

(20) Which of the given functions is even, which of the given functions is odd, and which of the given functions is neither? Explain carefully.

\[
\begin{align*}
  f(x) &= x^4\sqrt{1+x^2} \\
  g(x) &= x^3 + 1 \\
  h(x) &= x\sqrt{1+x^2}
\end{align*}
\]

The function \( f(x) \) is even because

\[
f(-x) = (-x)^4\sqrt{1+(-x)^2} = x^4\sqrt{1+x^2} = f(x)
\]

The function \( h(x) \) is odd because

\[
h(-x) = (-x)\sqrt{1+(-x)^2} = -(x\sqrt{1+x^2}) = -h(x)
\]

The function \( g(x) \) is neither because

\[
g(-x) = (-x)^3 + 1 = -x^3 + 1 \neq x^3 + 1 = g(x)
\]

and

\[
g(-x) = (-x)^3 + 1 = -x^3 + 1 \neq -(x^3 + 1) = -g(x)
\]

(21) Find functions \( f(x) \) and \( g(x) \) such that \( f(x) \) is even, \( g(x) \) is odd and \( f(x) + g(x) = 5x^5 - 7x^4 - 5x^3 + 8x^2 - x + 10 \).

We have \( f(x) = -7x^4 + 8x^2 + 10 \) and \( g(x) = 5x^5 - 5x^3 - x \). The function \( f(x) \) is even because

\[
f(-x) = -7(-x)^4 + 8(-x)^2 + 10 = -7x^4 + 8x^2 + 10 = f(x)
\]

The function \( g(x) \) is odd because

\[
g(-x) = 5(-x)^5 - 5(-x)^3 - (-x) = -(5x^5 - 5x^3 - x) = -g(x)
\]

(22) Express the function \( f(x) = \sqrt{1 + \cos^2 x} \) as the composition of three simpler functions.

If \( f_1(x) = \cos x \), \( f_2(x) = 1 + x^2 \) and \( f_3(x) = \sqrt{x} \) then \( f(x) = f_3(f_2(f_1(x))) \).